Solutions in Integers for the Quadratic Diophantine Equation $w^2 - 6z^2 + 8w - 24z - 24 = 0$

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Abstract— The polynomial solutions of Diophantine equation $D: w^2 - 6z^2 + 8w - 24z - 24 = 0$ is considered in $\mathbb{Z}$(x). A few recurrence relations and some formulae among the solutions (w_n, z_n) is also discussed in detail.

Index Terms— Diophantine equation, Polynomial solution, Pell's equation, Continued Fraction Expansion.

1 Introduction

Number Theory deals with various aspects of Diophantine analysis, a subject which can be described briefly by saying that a great part of it is concerned with the discussion of the rational or integer solutions of polynomial equation $f(x_1, x_2, .. x_n) = 0$ with integer coeffcients. Quadratic Diophantine equations form a major part of research by the variety of problems. There are a few Diophantine equations in which the complete solution is known. For example, the pythagorean equation $x^2 + y^2 = z^2$ and the Pell's equation $x^2 = dy^2 + 1$. There is no universal method to solve these type of equations.

In this paper, we investigate positive integral solutions of the Quadratic Diophantine equation $w^2 - 6z^2 + 8w - 24z - 24 = 0$. The method involves reduction to Pell's equation and the concepts of continued fraction.

2 The Diophantine Equation $w^2 - 6z^2 + 8w - 24z - 24 = 0$

Consider the Diophantine equation

$$D: w^2 - 6z^2 + 8w - 24z - 24 = 0 \quad (1)$$

to be solved over $\mathbb{Z}$. The equation (1) as given above does not follow the usual method in order to find its solutions. The approach is to have a linear transformation T to the equation (1) and transform to a simpler form for which we can determine the integral solutions.

Let

$$T: \begin{cases} w = x + h \\ z = y + k \end{cases} \quad (2)$$

be the transformation where $h, k \in \mathbb{Z}$. Moving ahead, we apply T to D; we get

$$T(D): (x + h)^2 - 6(y + k)^2 + 8(x + h) - 24(y + k) = 24 \quad (3)$$

Equating the co-efficients of x and y to 0, we get $h = -4$ and $k = -2$. Hence for $w = (x - 4)$ and $z = (y - 2)$, we have the Diophantine equation as

$$\widetilde{D}: x^2 - 6y^2 = 16 \quad (4)$$

which is a Pell's equation. This aids us to find all the integer solutions (x_n, y_n) of $\widetilde{D}$ and then re-transfer all the results from $\widetilde{D}$ to D by using the inverse of T. Consider the most general Pell's equation

$$x^2 - 6y^2 = 1 \quad (5)$$

2.1 Theorem 2.1

Let $\widetilde{D}$ be the Diophantine equation in (4), then

1. The continued fraction expansion of $\sqrt{6} = [2:\overline{2,4}]$
2. The fundamental solution of $x^2 - 6y^2 = 1$ is
$$(u_1, v_1) = (5, 2)$$
3. For n ≥ 4,
$$u_n = 11(u_{n-1} - u_{n-2}) + u_{n-3}$$
$$v_n = 11(v_{n-1} - v_{n-2}) + v_{n-3}$$

Proof:

1. The continued fraction expansion of

$$\sqrt{6} = 2 + (\sqrt{6} - 2)$$

$$= 2 + \frac{1}{\frac{1}{\sqrt{6}-2}}$$

$$= 2 + \frac{1}{\frac{\sqrt{6}+2}{2}}$$

$$= 2 + \frac{1}{2 + \frac{\sqrt{6}-2}{2}}$$

$$= 2 + \frac{1}{2 + \frac{1}{4 + \frac{1}{\sqrt{6}-2}}}$$

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Therefore, the continued fraction expansion of $\sqrt{6}$ is $[2: \overline{2,4}]$

2. It is easily seen that $(u_1, v_1) = (5, 2)$ is a solution of $x^2 - 6y^2 = 1$

Since $x^2 - 6y^2 = 25 - 6 \times 4 = 25 - 24 = 1$

3. If $(u_1, v_1) = (5, 2)$ is the fundamental solution of $x^2 - 6y^2 = 1$, then the other solutions (u_n, v_n) of $x^2 - 6y^2 = 1$ can be derived by using the equalities $(u_n + v_n\sqrt{6}) = (u_1 + v_1\sqrt{6})^n$ for n ≥ 2, as

$$\begin{pmatrix} u_n \\ v_n \end{pmatrix} = \begin{pmatrix} u_1 & 6v_1 \\ 2 & u_1 \end{pmatrix}^n \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Therefore, it can be shown by mathematical induction on 'n' that, the solution set is

n	0	1	2	3	4	5
u_i	5	49	485	4801	47525	490449
v_i	2	20	198	1960	19402	192060

The solution set satisfies the recurrence relations

$$u_n = 11(u_{n-1} - u_{n-2}) + u_{n-3}$$
$$v_n = 11(v_{n-1} - v_{n-2}) + v_{n-3}$$

Note that we denote the integer solutions of $x^2 - 6y^2 = 16$ by (x_n, y_n) and the integer solution of $x^2 - 6y^2 = 1$ by (u_n, v_n). Then we have the following theorem.

THEOREM 2.2

Define the sequence (x_n, y_n) of positive integers by (x_1, y_1) = (20,8) and

$$x_n = 20u_{n-1} + 48v_{n-1} \quad (6)$$
$$y_n = 8u_{n-1} + 20v_{n-1} \quad (7)$$

where (u_n, v_n) is a sequence of positive solutions of $x^2 - 6y^2 = 1$, then

1. (x_n, y_n) is a solution of $x^2 - 6y^2 = 16$ for any integer n ≥ 1.
2. For n ≥ 2

$$x_{n+1} = 5x_n + 12y_n$$
$$y_{n+1} = 2x_n + 5y_n$$

3. For n ≥ 4

$$x_n = 11(x_{n-1} - x_{n-2}) + x_{n-3}$$
$$y_n = 11(y_{n-1} - y_{n-2}) + y_{n-3}$$

Proof:

1. It is seen that $(x_1, y_1) = (20,8)$ is a solution of $x^2 - 6y^2 = 16$.
   Since
   $x_1^2 - 6y_1^2 = (20)^2 - 6(8)^2 = 400 - 384 = 16$
2. Let (x_n, y_n) be the integer solution of $x^2 - 6y^2 = 16$ and (u_n, v_n) be the integer solution of $x^2 - 6y^2 = 1$. We have

$$(x_{n+1} + y_{n+1}\sqrt{d}) = (u_1 + v_1\sqrt{d})^n(x_1 + y_1\sqrt{d}) \quad (8)$$

The value $n = 1$ in (8) yields $x_2 + \sqrt{6}y_2 = 196 + 80\sqrt{6}$ which gives (196, 80) as the second solution of equation (4).

Expanding equation (8) and equating the rational and irrational coefficients, we obtain the relation connecting (4) and (5) as

$$x_n = 20u_{n-1} + 48v_{n-1}$$
$$y_n = 8u_{n-1} + 20v_{n-1} \quad (9)$$

We see that,

$$\begin{aligned} x_n^2 - 6y_n^2 &= (20u_{n-1} + 48v_{n-1})^2 - 6(8u_{n-1} + 20v_{n-1})^2 \\ &= 16u_{n-1}^2 - 96v_{n-1}^2 \\ &= 16(u_{n-1}^2 - 6_{n-1}^2) \\ &= 16 \times 1 \\ &= 16 \end{aligned}$$

Therefore, (x_n, y_n) given by (9) is a solution of $x^2 - 6y^2 = 16$. Applying Brahmagupta's lemma between the solutions of (4) and (5), we get $x_{n+1} = u_1x_n + v_1y_nd$ and $y_{n+1} = v_1x_n + u_1y_n$ Since $u_1 = 5$ and $v_1 = 2$.

$$x_{n+1} = 5x_n + 12y_n$$
$$y_{n+1} = 2x_n + 5y_n \quad (10)$$

This proves 2.

3. For different values of n, we get the solution set of equation (4) which is expressed as a table given below.

n	0	1	2	3	4
x_n	20	196	1940	19204	190100
y_n	8	80	792	7840	77608

For n ≥ 4, it can be shown that the set of solutions of equation (4) satisfies the recurrence relations.

$$x_n = 11(x_{n-1} - x_{n-2}) + x_{n-3}$$
$$y_n = 11(y_{n-1} - y_{n-2}) + y_{n-3} \quad (11)$$

We proved that (x_1, y_1) = (20, 8) is the fundamental solution of $\widetilde{D}$. Also we showed that $h = -4$ and $k = -2$, so the base of T is $T[h, k] = \{-4, -2\}$ as we had claimed. The Diophantine equation D could be transformed into the Diophantine equation $\widetilde{D}$ by the transformation T, $w = x - 4$ and $z = y - 2$.

THEOREM 2.3

Let D be the Diophantine equation in (1) then

1. The fundamental solution of D is (w_1, z_1) = (16,6).
2. Define the sequence $\{((w_n, z_n)\}\, n \geq 1 = \{(x_n - 4, y_n - 2)\}$ where $\{(x_n, y_n)\}$ defined in (10). Then (w_n, z_n) is a solution of D. So it has infinitely many solutions $(w_n, z_n) \in \mathbb{Z} X \mathbb{Z}$
3. The solution (w_n, z_n) satisfy

$$w_n = 5w_{n-1} + 12z_{n-1} + 40$$
$$z_n = 2w_{n-1} + 5z_{n-1} + 16$$

4. The solution (w_n, z_n) satisfy the recurrence relations

$$w_n = 11(w_{n-1} - w_{n-2}) + w_{n-3}$$
$$z_n = 11(z_{n-1} - z_{n-2}) + z_{n-3}$$

Proof:

1. It is easily seen that (w_1, z_1) = (16, 6) is the fundamental solution of D since $16^2 - 6(6)^2 + 8(16) - 24(6) = 24$
2. We prove by mathematical induction

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Let $n = 1$, then $(w_1, z_1) = (x_1 - 4, y_1 - 2) = (16, 6)$ which is the fundamental solution and so is a solution of D. Let us assume that the Diophantine equation in (1) is satisfied for $(n-1)$, that is

$(x_{n-1} - 4)^2 - 6(y_{n-1} - 2)^2 + 8(x_{n-1} - 4) - 24(y_{n-1} - 2) - 24 = 0$

We want to show that this equation is also satisfied for n.

$$\begin{aligned} &w_n^2 - 6z_n^2 + 8w_n - 24z_n - 24 \\ &= (x_n - 4)^2 - 6(y_n - 2)^2 + 8(x_n - 4) - 24(y_n - 2) - 24 \\ &= x_n^2 - 6y_n^2 - 16 \\ &= 0 \end{aligned}$$

Since (x_n, y_n) is a solution of $\widetilde{D}$; from (2), we get $(w_n, z_n) = (x_n - 4, y_n - 2)$ is a solution of D.

3. Using (10) on

$$x_n = 5x_{n-1} + 12y_{n-1}$$
$$y_n = 27x_{n-1} + 5y_{n-1}$$

Adding - 4 on both sides, we get

$x_n - 4 = 5x_{n-1} + 12y_{n-1} - 4$

Adding -2 on both sides, we get

$y_n - 2 = 2x_{n-1} + 5y_{n-1} - 2$

We know that $w_n = x_n - 4$ and $z_n = y_n - 2$

Therefore, $x_n = w_n + 4$ and $y_n = z_n + 2$

$x_n - 4 = 5x_{n-1} + 12y_{n-1} - 4$

$(w_n + 4) - 4 = 5(w_{n-1} + 4) + 12(z_{n-1} + 2) - 4$

Therefore we get

$$w_n = 5\,w_{n-1} + 12z_{n-1} + 40 \qquad (12)$$

Similarly, we get

$$z_n = 2w_{n-1} + 5z_{n-1} + 16 \qquad (13)$$

4. The following table represents the solution set of (12) and (13)

n	0	1	2	3
w_n	16	192	1936	19200
z_n	6	78	790	7836

We can see that for n = 4

$$\begin{aligned} w_4 &= 11(w_3 - w_2) + w_1 \\ &= 11(19200 - 1936) + 192 \\ &= 190096 \end{aligned}$$

Let us assume that this relation is satisfied for $(n-1)$

$$w_{n-1} = 11(w_{n-2} - w_{n-3}) + w_{n-4} \qquad (14)$$

Then applying the previous assertion, equation (12) and equation (14) we can conclude that $w_n = 11(w_{n-1} - w_{n-2}) + w_{n-3}$ for n ≥ 4.

Now we can prove that z_n also satisfies the recurrence relation. For n = 4 we get

$$z_4 = 11(z_3 - z_2) + z_1 = 11(7838 - 790) + 78 = 77606$$

Let us assume that this relation is satisfied for $(n-1)$

$$z_{n-1} = 11(z_{n-2} - z_{n-3}) + z_{n-4} \qquad (15)$$

Then applying the previous assertion by equation (13) and equation (15) we can conclude that $z_n = 11(z_{n-1} - z_{n-2}) + z_{n-3}$ for n ≥ 4.

3 Conclusion:

Diophantine Equations are one of many wonders in the world of Mathematics. They uniquely stand out and beckons mathematicians all over the world to share the richness of the computational world. Unravelling its mysteries, these equations have proved beyond doubt that there is no defined method to find the solutions. Although, it looks simple and easy; it sure takes time and effort to strive to get a possible solution to these mind-boggling problems.

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